Characterising Classes of c.e. Turing degrees using strong reducibilities from above

George Barmpalias, Rod Downey, Noam Greenberg

14th December 2007

Computable approximations

Definition

A uniformly computable sequence $\langle f_s \rangle$ is a computable approximation for a function f if f_s converges to f pointwise (in the discrete topology).

That is, for every x,

$$m_{\langle f_s \rangle}(x) = \#\{s : f_{s+1}(x) \neq f_s(x)\}$$

is finite for all x, with ultimate value $f(x)$.

Theorem (Schoenfield)

A function f has a computable approximation iff $f \leqslant_T \mathbf{0}'$.

Weak truth-table reducibility

Definition

A reduction $\Gamma(A) = B$ is a weak truth-table reduction if its use is bounded by some computable function.

ω-C.E. FUNCTIONS

DEFINITION

A function is ω-c.e. if it has some computable approximation $\langle f_s \rangle$ such that $m_{\langle f_s \rangle}$ is bounded by some computable function.

FACT

A function f is ω-c.e. iff $f \leqslant_{wtt} \mathbf{0}'$.

Totally ω-c.e. degrees

Definition

A c.e. Turing degree $\mathbf{d}$ is totally ω-c.e. if every $f \leqslant_T \mathbf{d}$ is ω-c.e.

Theorem (D,G, Weber)

A c.e. degree is totally ω-c.e. iff it doesn't bound a critical triple in the c.e. degrees below it.

The proof uses *permitting* and *anti-permitting* arguments.

Ranked sets

Definition

A set is ranked if it is an element of some countable effectively closed (Π^0_1) class.

Theorem (Chisholm, Chubb, Harizanov, Hirschfeldt, Jockusch, McNicholl and Pingrey)

If $\mathbf{d}$ *is c.e. and not totaly* ω*-c.e., then there is some c.e.* $A \in \mathbf{d}$ *which is not wtt-reducible to any ranked set.*

TOTALLY ω-C.E. DEGREES AND WTT REDUCTIONS

THEOREM

The following are equivalent for a c.e. degree $\mathbf{d}$*:*

1. *Every set in* $\mathbf{d}$ *is wtt-reducible to a ranked set.*
2. *Every set in* $\mathbf{d}$ *is wtt-reducible to a hypersimple set.*
3. *Every set in* $\mathbf{d}$ *is wtt-reducible to a proper initial segment of a computable, scattered linear ordering.*
4. $\mathbf{d}$ *is totally* ω*-c.e.*

Moreover, the equivalence still holds if in any of (1), (2) or (3), "set" is replaced by "c.e. set".

Array recursive degrees

Definition

A c.e. degree $\mathbf{d}$ is uniformly totally ω-c.e. if there is a computable function h such that every $f \leqslant_T \mathbf{d}$ has a computable approximation $\langle f_s \rangle$ such that $m_{\langle f_s \rangle}$ is bounded by h.

Fact

A c.e. degree is uniformly totally ω-c.e. iff it is array recursive.

COMPUTABLE LIPSCHITZ REDUCTIONS

DEFINITION

A reduction $\Gamma(A) = B$ is a computable Lipschitz reduction if its use is bounded by $n + c$ for some constant c.

Array recursive degrees and cl reductions

Theorem

The following are equivalent for a c.e. degree **d***:*

1. *There are left-c.e. reals* $\alpha_0, \alpha_1 \in \mathbf{d}$ *which have no common upper bound in the cl-degrees of left-c.e. reals.*
2. *There is a left-c.e. real* $\alpha \in \mathbf{d}$ *which is not cl-reducible to any random left-c.e. real.*
3. *There is a set* $A \in \mathbf{d}$ *which is not cl-reducible to any random left-c.e. real.*
4. **d** *is array non-recursive.*

What about randoms in general?

Theorem

1. *If* **d** *is non* GL_2*, then* **d** *computes some* A *which is not cl-reducible to any random real.*
2. *If* **d** *is c.e. traceable, then every* $A \leqslant_T$ **d** *is cl-reducible to a random real.*

This leaves a gap, even in the c.e. degrees.